FOSTERING CLASSROOM COMMUNICATION ON REPRESENTATIONS OF FUNCTIONS

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Teaching students how to use and interpret various representations of functions remains an enduring challenge for educators. Providing students access to such representations is a key feature of providing them access to mathematical participation and communication. This study is based on a teaching experiment, which aims to provide students access to the discourse on representations of functions by eliciting students' discourses and making them explicit topics of reflection in a post-secondary classroom. The results indicate that the pedagogical approach used in this study has the potential to foster mathematical communication in the classroom as evidenced by students' awareness of the tacit aspects of their discourses that shape their thinking about representations of functions.

Keywords: Advanced Mathematical Thinking; Classroom Discourse; Instructional Activities and Practices; Post-Secondary Education

Representations mediate thinking about mathematical concepts and play critical roles in mathematical communication. Kaput and Rochelle (1999) argue that the emergence of new representational forms can help learners engage in powerful mathematical ideas that can otherwise remain inaccessible to them—an affordance they refer to as *democratization of access*. Arcavi (2003) also notes that visual representations can help students think about concepts and meanings which can be bypassed by symbolic representations. The representational system of a concept carries the underlying structure of that concept and the possible irrelevancies within a representational system "are dismissed or unnoticed by experts" (Arcavi, 2003, p. 232). For students who cannot see the underlying structure of a concept within a representational system, those irrelevancies can be significant since experts of mathematical discourse can "lose the ability to see as different what children cannot see as the same" (Sfard, 2008, p. 59). These arguments indicate that mathematical representations have the potential to provide learners access to mathematical communication. However, if their roles and use remain invisible to the students, they can also lead to miscommunication in the classrooms. Teachers play critical roles in making mathematical ideas transparent for students to enhance mathematical communication. This work is based on a teaching experiment that aims to provide students access to the discourse on representations of functions and explores whether this pedagogical approach has the potential to foster classroom communication.

Function is a central concept in K-12 and undergraduate mathematics and it is a challenging topic to learn due to the various notions associated with the concept (Eisenberg, 1991). In particular, many researchers argue that students have difficulties moving flexibly across graphical, algebraic, tabular, and verbal representations of functions (Monk, 1994; Schoenfeld, Smith & Arcavi, 1993; Sierpinska, 1992; Tall, 1996). Teaching students how to use and interpret various representations of functions remains an enduring challenge for educators and mathematics education researchers. Given the role functional relationships and representations play in mathematics and science as well as everyday interpretations of data, students' lack of access to the discourse on representations of functions can hinder their access to mathematical participation and communication.

This study is based on a teaching experiment that used a discursive approach to elicit students' discourses about representations of functions and made them explicit topics of discussion and reflection in the classroom. The goal was to teach students how various representations of the

function concept are similar to and different from each other to address particular aspects of functional representations that can remain implicit for the students. The study addresses the following questions: What are the features of a discursive teaching approach that aims to provide learners access to the discourse on representations of functions and how can this approach foster communication in the classroom about representations of functions?

Theoretical Framework

This work aligns with the theoretical approaches that view learning as becoming a more fluent participant in mathematical communities of practice. Each community of practice leaves a historical trace of physical, linguistic, and symbolic artifacts as well as social structures that define the characteristics of participation (Lave & Wenger, 1991). From this perspective, learning to become a fluent participant in mathematical communities of practice involves learning to speak mathematically and learning to use the artifacts of the practice in the manner of full participants by engaging in mathematical activities. Lave and Wenger (1991) refer to the artifacts employed in any practice as *technology of practice* and argue that transparency of the technologies of practice with respect to their meaning and use is a critical condition for access. The visual representations that mediate mathematical communication (e.g., graphs, symbols, words) are among the technologies of mathematical practice. When the meaning and use of these representations are commonly agreed upon, they enhance mathematical communication. If their use and purposes remain invisible for the learners, then they may also hinder communication in the classrooms. For example, Güçler's (2014) earlier work demonstrates that merely presenting a mathematical representation to students is not sufficient for transparent communication in the classroom.

This study views mathematics as a discourse, where discourse refers to "different types of communication set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (Sfard, 2008, p.93). From this lens, providing access to the technologies of mathematical practice is tantamount to providing access to the *discourse* on those technologies. Sfard (2008) uses the term *meta-level rules* to refer to the elements of mathematical discourse that can remain implicit for learners and separates them from *object-level rules*. Object-level rules are about the behavior of the objects of mathematical discourse, whereas meta-level rules characterize the patterns in the activity of participants. Meta-level rules are "about the actions of the discursants, not about the behavior of mathematical objects" (Sfard, 2008, p. 201). For example, "the graph of the function $y = x^2$ is a parabola" is an object-level rule of mathematical discourse whereas the patterns in learners' actions when drawing that graph (e.g., using the assumption of continuity consistently when thinking about functions and their representations) constitute the meta-level rules in the learners' discourses. The meta-level rules of mathematical discourse are often tacit and, if not made explicit, learners can talk about the same mathematical object (e.g., a graph) in different ways, leading to possible miscommunication (Güçler, 2013).

Providing learners access to the technologies of mathematical practice requires the teacher to attend to the tacit aspects of their meaning and use in the context of the classroom. For this study, the tacit aspects of the technologies of practice refer to the meta-level rules in participants' discourses that shape their thinking about representations of functions.

Methodology

This work is part of a larger study that explored student thinking on functions, limits, derivatives, and integrals over the course of 13 weeks. The focus here is on the classroom discussions about representations of functions that took place during the first 3 weeks. The study followed a teaching experiment methodology as outlined by Steffe and Thompson (2000), which involves experimentation with the methods that can influence student thinking. This paper is on the features of

the teaching experiment and the nature of discourse it elicited in the classroom regarding representations of functions. The participants were one pre-service and seven in-service high school teachers, hereon referred to as *the students*, taking a mathematics content course on calculus for their teaching licensure programs. The researcher was the instructor of the course and all of the students taking the course volunteered to participate in the study. The classroom sessions about functions were video-taped. The sections during which the instructor and students talked about representations of functions were transcribed. The transcripts included the utterances and actions of the participants.

Consistent with the theoretical assumptions of the study, a specific goal of the teaching experiment was to make the tacit meta-level rules in learners' discourses about representations of functions explicit topics of discussion and reflection in the classroom. In order to do that, it was important to bring forward the various ways in which learners used and talked about representations of functions. The activities on representations of functions were designed so that the students had the potential to act according to different meta-level rules, leading to different realizations of those representations and the function concept. Those instances were considered critical interms of eliciting students' existing discourses on representations of functions and examining which aspects of them were visible or invisible for the learners.There were also specific discussions about the similarities and differences among various representations of functions with respect to the different meta-level rules on which they are based. At the end of each activity, after eliciting the students' discourses, the instructor explicated the meta-level rules in their discourses. This was an intentional part of the teaching experiment with the goal of making transparent the different meanings and uses of representations of functions andproviding students access to these technologies of mathematical practice. Throughout the three lessons on functions, students worked on various activities on representations of functions, two of which will be presented in the next section.

While the visuals students used constituted the representations in their discourses on functions, *how* they used those representations (their discursive acts when visualizing functions) revealed the meta-level rules in their discourses. For the analysis of the meta-level rules in students' discourses, particular attention was given to the types of assumptions students used when thinking about representations of functions (e.g., assumption of continuity, regularity, discreteness). In the next section, the discussions on some of the classroom activities about representations of functions are presented with a particular focus on the meta-level rules in students' discourses. Students' awareness of the tacit meta-level rules shaping their discourses about the function concept and its representations were considered as indicators of enhanced classroom communication. All the student names used in the study are pseudonyms.

Results

The first classroom discussion about representations of functions took place during the first lesson on functions when students were asked to provide a definition of the concept in their own words. When multiple students mentioned *graph* as a definition of function, the instructor hypothesized that they may be using the assumption that a function is the same thing as its representation. She posed the following question to initiate a discussion: "If a function *is* a graph, then is a graph a function?" The students mentioned that "a graph is not always a function" but did not realize that they were using *function* and *graph* as equivalent words in their initial definitions. The instructor then asked "If a function *is* a graph, then can we also define a function as a table or algebraic expression?" The responses indicated that the students were comfortable with defining a function "as a graph or equation" but "not as a table". When asked to elaborate, they said that "a table is only a representation". The teacher asked whether, and how, a graph or algebraic expression was different than a table. The students quickly realized that all of those visuals were representations of functions. They were confused with their acceptance of some of those representations as a definition of function and rejection of others as signifying a definition. To help students resolve this

conflict, the instructor asked if a function is the same thing as its representation. The students said they “always assumed continuity”, which made it “natural [for them] to think about graphs and equations as functions”. They also mentioned using tables to draw graphs of functions but they did not consider a table as a function because it was “just a set of values”.

Although this discussion was about definitions of functions, it revealed some meta-level rules that shaped students’ discourses about representations of functions as well. The students were using the assumption of continuity as a meta-level rule when thinking about a function as a graph or equation. The discussion revealed that they were also thinking about that equation as a single rule through the assumption of regularity. The students used the assumption of discreteness when talking about a table as a set of values and they considered a table as a tool to generate a graph. However, they did not realize that the meta-level rule in their discourses when talking about the tabular representations of functions (using the assumption of discreteness) was not compatible with the meta-level rule in their discourses when talking about graphs and algebraic equations (using the assumption of continuity). At that point, using the ideas students generated during the discussion, the instructor explicated the tacit meta-level rules shaping their thinking about functions and their representations (e.g., students’ referral to a function as the same thing as one of its representations; their incompatible assumptions when transitioning from one representation to another).

The students worked on another activity on representations of functions during the second lesson on functions. They were given a tabular representation as shown in Table 1 and were asked “what can you say about *F(x)* based on this representation?”

Table 1: Tabular representation of *F(x)*

x	*F(x)*
-3	3
-2	2
-1	1
0	0
1	1
2	2
3	3

Carrie initially mentioned that it should be the absolute value function, which she represented algebraically as $y = |x|$. Realizing that Carrie was thinking about *F(x)* through the assumption of regularity, the teacher asked students how they would translate this tabular representation to a graphical one. Fred said the prior class discussions made him think that he needed to know the domain on which the function was restricted. He argued that the algebraic representation of the function should also include $x \in \{-3,-2,-1,0,1,2,3\}$ and said the graph should consist of “a set of points” as shown in Figure 1(a). All of the students agreed that “the function” should be represented discretely as a set of points on the Cartesian plane. Since the students automatically assumed that *F(x)*is a function, the teacher then asked what would happen if the domain of *F(x)* in Table 1was $[-3,3]$. Figure 1(b) shows the graph students generated for the question.

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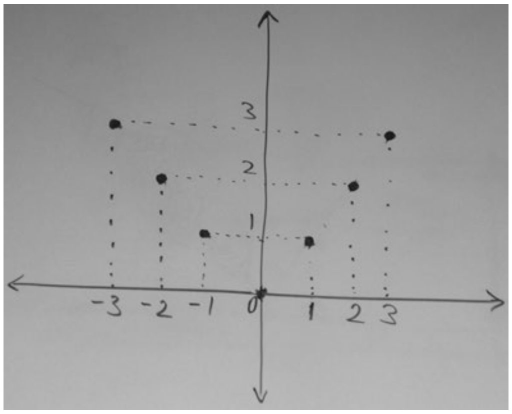

Figure 1(a): The first graph students generated during the activity

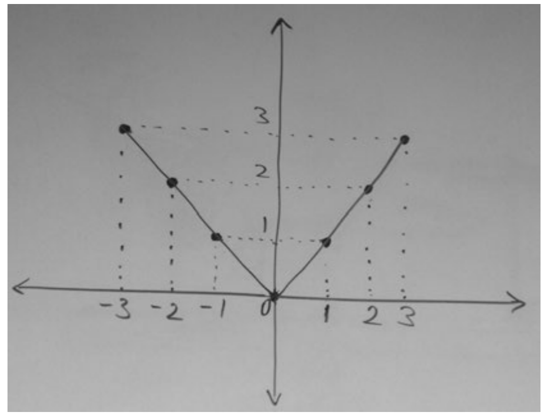

Figure 1(b): The second graph students generated during the activity

Note that, despite their realizations of the importance of the domain of a function based on previous classroom discussions, the students were still using the assumptionsof continuity and regularity as meta-level rules in their discourses when they generated the graph in Figure 1(b). They were thinking about *F(x)* as the continuous absolute value function over the interval $[-3,3]$, possibly due to the pattern they saw in the tabular representation in Table 1. Although the students recognized the tabular representation as consisting of static set of points using the assumption of discreteness when they generated Figure 1(a), they were using the assumption of continuity when they generated the graph in Figure 1(b). This clash in the utilization of different meta-level rules students used prompted the teacher to ask "how do you know that *F(x)* is a function?" In response to the students' puzzled looks, she drew two graphs as shown in Figure 2(a,b), which satisfied the conditions in Table 1 and asked students to elaborate on those graphs.

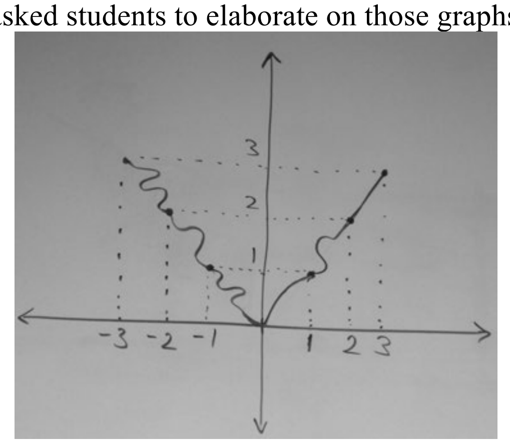

Figure 2(a): The first graph the teacher generated during the activity

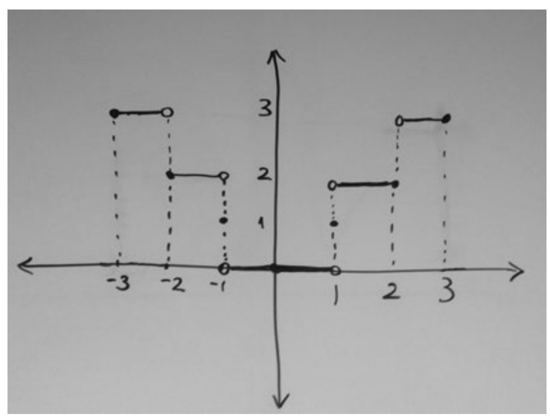

Figure 2(b): The second graph the teacher generated during the activity

The students quickly realized that the graph in Figure 2(a) did *not* represent a function on $[-3,3]$ and the one in Figure 2(b) was not continuous although both graphical representations were consistent with the set of values in Table 1. In addition, the students realized that Figure 2 (b) "does not represent a regular function that has a single rule". Sally mentioned that these challenges were occurring because they "only worked with continuous functions" in their education. Steve then mentioned that he used tabular representations every time he modeled continuous real-life phenomena. This led to discussions about how using different assumptions shape thinking about functions and their representations. During those discussions, the students explicitly mentioned that

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they were using two different assumptions—continuity and discreteness—which indicated that they became aware of those meta-level rules. At the end of the classroom discussion, the teacher explicated the connections and differences between a definition of function and the visual mediators that represent the concept. While doing so, she used students' considerations of function as a graph and rejection of any graph as a function to encourage them to think about how definitions of functions are formulated to avoid ambiguity.

The features of the teaching approach used during all the activities about representations of functions were consistent with those demonstrated in the aforementioned activities. Those features included (a) eliciting how students talk about functions and their representations, (b) listening to their responses carefully to capitalize on the instances in which students reveal the meta-level rules in their discourses by asking probing questions, (c) creating opportunities so that students act according to different meta-level rules, leading to communicational conflicts, (d) giving students opportunities to reflect on their discourses to resolve those conflicts, and (e) explicating the emerging meta-level rules in their discourses at the end of the discussions for further reflection.

The results of the study provide some evidence how a discursive approach to teaching has the potential to foster classroom communication, particularly with respect to students' awareness of the meta-level and tacit aspects of their discourses on functions and their representations. Although the focus of this paper is mainly on the classroom discussions due to space constraints, additional evidence regarding the affordances of the teaching experiment in fostering classroom communication is given in Table 2. Such evidence is based on students' weekly journal entries about functions in which they further reflected on any aspect of the classroom discussions that they found interesting.

Table 2: Examples from students' reflections on representations of functions

Sally: I didn't imagine that functions would be so complicated, there is much more to it than we are used to seeing…I'm on the mindset that the representation of the function doesn't define the function.
Martin: One theme that keeps coming up…is the notion of continuity. I feel I will need to develop strategies to address this issue. Students, just like us, have a tendency to assume continuity despite not being told or shown that a function is continuous.
Lea: The class activity highlighted the idea that both graphs and tables are merely representations of a function, and cannot always depict all the possible values in a function… However, even though the graph and table are limited, they are both still important in understanding functions as they offer students a unique visual representation of a function… This example, along with others, also demonstrated that functions are very often discontinuous, or piecewise and may also contain more than one rule to it.
Ron: In the some ways, it makes sense to gradually introduce different representation of a function over time as well as redefine a function as new concepts are presented. Through our class discussions, we talked about 8 different ways to represent a function... I think the big question is when we should introduce set notation and synthetic representations of a function.

Table 2 includes some representative examples of students' reflections on representations of functions. By the end of the lessons on functions, these students were aware of some of the assumptions they used (e.g., continuity and regularity) as meta-level rules in their discourses on function (Table 2, [2], [3]). They also talked about the difference between the abstract concept of function and its representations (Table 2, [1], [3]). Further, since these students were also teachers,

the classroom discussions helped them think about how to teach these ideas to their students (Table 2, [2], [4]). These results suggest that the teaching approach used in the study helped students be aware of some of the meta-level rules shaping their discourses on representations of functions and the function concept. The study also confirms the tacit nature of meta-level rules and their role in mathematical communication since the students in the study mentioned that they never learned about those meta-level rules before this course.

Discussion

Representations of functions are among the technologies of mathematical practice and play important roles in classroom communication. However, if their meaning and roles are not shared by participants, they may also lead to miscommunication. This study demonstrated that some aspects of the discourse on such technologies of practice can remain invisible to learners—even to those who have been exposed to the technologies in their prior education. This finding is in accordance with Güçler's (2014) previous work and indicates that teachers should not take the communicative power of mathematical representations for granted; they need to make the discourse on those technologies of practice transparent for their students. The notion of transparency, which refers the visibility of the use and meaning of technologies of practice (Lave and Wenger, 1991), is a critical condition for providing learners access.

In this study, access to the technologies of practice was conceptualized as access to the discourse on those technologies through the use of Sfard's (2008) framework. This framework was useful in identifying and examining the features of classroom discourse that can remain tacit for the learners (e.g., meta-level rules) and served as a lens that helped shape the design of the teaching experiment used in this study. The results of the study indicate that a teaching approach that is responsive to the discourses of the students, which also elicits the meta-level aspects of participants' mathematical discourses, has the potential to foster classroom communication and transparency needed to make mathematical ideas clearer for the learners. Such a teaching approach elicits the various ways in which students think about technologies of mathematical practice to highlight which aspects of them remain invisible for the students in the context of the classroom.

This work may have some implications for teacher education. The participants of this study were pre-service and in-service high school teachers. Although they were considered as students in the content course they were taking, they were also the teachers of that content in high school classrooms. The results of the study indicate that these teachers' reflections on their discourses on functions and their representations also triggered their reflections on how to teach those ideas to their own students. Some of those teachers mentioned using the activities they worked on in the classroom with their own students. In this respect, this study offers educators some activities and ideas that may be useful in teacher education courses and professional development.

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